Exercise Sheet #7

Course Instructor: Ethan Ackelsberg
Teaching Assistant: Felipe Hernández

P1. Throughout this problem we denote λ as the Lebesgue measure on $\mathbb{R}$.

(a) Let A be a Lebesgue-measurable set on the real line such that $\lambda(A) > 0$. Show that the difference set $A - A = \{x - y \mid x, y \in A\}$ contains an open neighborhood of 0 in $\mathbb{R}$.
Hint: Prove that for each $r \in (1/2, 1)$, there is an interval $(a, b) \subseteq \mathbb{R}$ such that $\lambda(A \cap (a, b))/(b - a) \geq r$.

(b) Let $(H, +)$ be a Lebegue measurable proper subgroup of $(\mathbb{R}, +)$. Show that $\lambda(H) = 0$.

P2. **(a)** Show that the Dirac functional $\delta_0 \in \mathcal{M}[0, 1]$ defined by $\delta_0(f) := f(0)$ is not of the form

$$\delta_0(f) = \int_0^1 f(t)g(t)\mathrm{d}t \quad (f \in C[0, 1])$$

for any $g \in C[0, 1]$.

(b) Define $\psi : C[0, 1] \to \mathbb{R}$ by

$$\psi(f) = \frac{f(0) + f(1)}{2} + \int_0^1 tf(t)\mathrm{d}t.$$

Determine the measure from the Riesz-Markov-Kakutani theorem corresponding to ψ, i.e. a regular Borel measure μ on $[0, 1]$ such that $\psi(f) = \int_{[0,1]} f \, \mathrm{d}\mu$ for $f \in C[0, 1]$. Calculate $\mu([0, 1])$.

P3. In this exercise, we will construct a Haar measure[1] on the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. For this, recall that one can identify functions $f : \mathbb{T}^n \to \mathbb{C}$ with $\mathbb{Z}^n$-invariant functions $F : \mathbb{R}^n \to \mathbb{C}$ on $\mathbb{R}^n$ (i.e. we require $F(x + m) = F(x)$ for all $m \in \mathbb{Z}^n$). Furthermore, f is continuous (measurable) if and only if F is continuous (measurable). We define a measure m on $\mathbb{T}^n$ by requiring that

$$\int_{\mathbb{T}^n} f \, \mathrm{d}m = \int_{[0,1]^n} F \, \mathrm{d}m_{\mathbb{R}^n}$$

where $m_{\mathbb{R}^n}$ is the Lebesgue measure on $\mathbb{R}^n$ and f, F are measurable and correspond to each other. Justify that m is well define and show that m is a Haar measure on $\mathbb{T}^n$.

[1] A Haar measure is a Radon measure on a locally compact topological group $(G, +)$ that is left-invariant, meaning that for any Borel set S and $g \in G$, $\mu(g + S) = \mu(S)$.